$$x_1 - n \equiv n + k(x_2 - x_1) \mod m.$$

Hence
$$n \equiv \frac{x_1}{2} + k\frac{x_1 - x_2}{2} \mod m/2.$$

That is, n has $2m/d$ distinct values. These may always be represented as the vertices of a regular polygon of $2m/d$ sides, as will appear more clearly from what follows.

The preceding results may readily be extended so as to apply to the case when l has any arbitrary value. If d represents the highest common factor of m and all the differences of the form $x_\alpha - x_\beta$, the subtraction group generated by the reflections corresponding to the operations $x_1 - n$, $x_2 - n$, ..., $x_l - n$ is again the dihedral rotation group of order $2m/d$. This results almost directly from the preceding considerations and the following elementary facts: (1) Any finite number of rational translations taken with respect to a rational modulus generate a cyclic group whose order is the quotient obtained by dividing the rational modulus by the largest number which is contained an integral number of times in all the translations and the modulus. This cyclic group may be generated by translations (or rotations*) which are equal to this largest divisor leading to integral quotients. (2) Any reflection followed by a translation is a reflection on the point obtained by translating the original point of reflection through half the original translation. That is, a translation followed by a reflection, or vice versa, is an operation of period two, or any translation whatever is transformed into its inverse by any reflection.

From these general considerations it follows that the subtraction group for an arbitrary value of l is just as elementary as when $l=2$, and that the number of special values of n is still equal to the order of the corresponding group and consist of two sets such that all the numbers of a set are obtained from any number of the set by the operations of the group. It is also clear that x_1, x_2, ..., x_l, m could be any rational numbers instead of being integral but this is only an apparent generalization and hence nothing would have been gained by the assumption.

When x_1, x_2, ..., x_l, m represent complex integers while n is any ordinary complex number the preceding considerations remain practically unchanged. For the sake of simplicity we shall again consider the case when $l=2$ by itself. Let $x_1 = a_1 + b_1 i$, $x_2 = a_2 + b_2 i$, $m = m_1 + m_2 i$, $n = x_0 + y_0 i$. The real parts of $x_1 - n$, $x_2 - n$ are obtained by reflecting the real part of n on the lines $x = a_1/2$, $x = a_2/2$, respectively, while the coefficients of i in these remainders are obtained by reflecting y_0 on the lines $y = b_1/2$, $y = b_2/2$, respectively. The order of the translation generated by the two reflections $x_1 - n$, $x_2 - n$ is the least common multiple of m_1/d_1, m_2/d_2, where d_1 is the highest common factor of $a_2 - a_1$ and m_1 while d_2 has the same meaning with respect to $b_2 - b_1$ and m_2. If m' represents these

*When the group is represented on a circle, the translations are replaced by rotations. This is sometimes most convenient.

least common multiples the group generated by the two reflections $x_1 - n$, $x_2 - n$ is the dihedral rotation group of order $2m'$.

The generalization of the case when l is arbitrary is so similar to the case considered above that it seems unnecessary to give details. The only important difference consists of the fact that m' represents the least common multiple of of m_1/d_1, m_2/d_2, where d_1 and d_2 are the highest common factors, respectively, of all such difference as $a_\alpha - a_\beta$ and $b_\alpha - b_\beta$. The numbers of the special values of n is again equal to the order of the corresponding group. It is clear that every dihedral group (including the four group) may be represented in an infinite number of ways as a subtraction group. The stress should, however, not be laid upon the fact that subtraction furnishes such interesting illustrations of this important system of groups, but rather upon the fact that these groups give a deeper and far reaching meaning to the fundamental operation of subtraction.

STANFORD UNIVERSITY, *September, 1904.*

THE SINKING-FUND OF THE UNITED STATES.

By G. B. M. ZERR.

The public debt of the United States is being paid by the sinking-fund in the following manner. During each fiscal year a sum is paid equal to one per cent. of the principal of the current debt, plus a sum equal to the interest on the part of the debt already paid at the rate of interest the debt bears. If such a sinking-fund had been operated under the same law from the beginning, how long would it require to pay the public debt, if the rate of interest the debt draws is four per cent. per annum?

A very excellent solution of the above problem is given in the *Mathematical Magazine* for September, 1904, by Theodore L. DeLand, who employs the Calculus of Finite Differences.

As the great debt of the United States will fall due in a few years, and, as its payment, then, will have to be met by a long-time loan at a different rate, which will change the present sinking-fund, we believe that a simple algebraic solution of this national problem will be interesting to the readers of the MONTHLY.

Let p=principal of the public debt at the beginning; r=.01, the rate per annum on the current principal; R=.04, the rate of interest the debt draws per annum; n=number of years required to pay the debt.

Then rp=first payment;

$p(1-r)$=unpaid part of debt after first payment;

$rp(1-r)+Rrp=pr(1-r+R)$=second payment;

$pr(1+1-r+R)$=total paid;

$p[1-r-r(1-r+R)]$=unpaid part after second payment;

$pRr(1+1-r+R)+pr[1-r-r(1-r+R)]=pr(1-r+R)^2$=thirdpayment;

$pr[1+1-r+R+(1-r+R)^2]$=total paid;

$p[1-r-r(1-r+R)-r(1-r+R)^2]$=unpaid part after third payment;

$pr[R+R(1-r+R)+R(1-r+R)^2]+pr[1-r-r(1-r+R)-r(1-r+R)^2]$
$=pr(1-r+R)^3$=fourth payment.

Similarly, $pr(1-r+R)^{n-1}$=nth payment. Hence

$$pr+pr(1-r+R)+pr(1-r+R)^2+pr(1-r+R)^3+\ldots\ldots+pr(1-r+R)^{n-1}=p.$$

$$\therefore pr[1+(1-r+R)+(1-r+R)^2+(1-r+R)^3+\ldots\ldots+(1-r+R)^{n-1}]=p.$$

$$\therefore r[(1-r+R)^n-1]/(R-r)=1.$$

$$\therefore (1-r+R)^n=R/r, \quad n=\frac{\log(R/r)}{\log(1-r+R)}.$$

$$n=\frac{\log 4}{\log(1.03)}=\frac{.6020599913}{.0128372247}=46.8995 \text{ years}=46 \text{ years, } 10 \text{ months, } 24 \text{ days.}$$

A PROPERTY OF THE GROUP $G_{2^{2n}}$ ALL OF WHOSE OPERATORS EXCEPT IDENTITY ARE OF PERIOD 2.

By L. E. DICKSON.

1. G is a commutative group since $ab=(ab)^{-1}=b^{-1}a^{-1}=ba$. As a concrete form of G we may take the group of the linear substitutions which multiply each of the $2n$ variables by ± 1.

It is always possible to separate the operators other than I of $G_{2^{2n}}$ into 2^n+1 sets each of 2^n-1 operators such that those of any set together with I form a group of order 2^n, and such that no two sets have a common operator. We consider the number N_n and character of all such separations into sets. Evidently $N_1=3$. We show that $N_2=56$, $N_3=2^{12}.3.5.31$.

2. Let first* $n=2$. The first set (a, b, ab) may be chosen in $\frac{1}{6}.15.14=35$ ways. The second set (A, B, AB) may then be chosen in $\frac{1}{6}.12.8$ ways, since A may be any operator except I, a, b, ab, and B any operator except these four and their products† by A. Then AB differs from I, a, b, ab, A, B. Indeed, a rectangular table of the operators of G_{16} is given by

*Cf. Ex. 2, p. 60, Burnside's *Theory of Groups*; errata, p. xvi.

†If, for example, $B=bA$, then $AB=b$ would occur in the first set.

$$(1)\qquad \begin{array}{cccc} I & a & b & ab \\ A & aA & bA & abA \\ B & aB & bB & abB \\ AB & aAB & bAB & abAB \end{array}$$

Each operator of the third set (a_3, b_3, a_3b_3) must occur in the minor M of the element I of the determinant of (1). Hence $a_3=cC, b_3=dD, a_3b_3=cdCD$, where c, d, cd form a permutation of a, b, ab, and C, D, CD a permutation of A, B, AB. Hence the operators a_3, b_3, a_3b_3 must be chosen from one *term* of the expansion of M. Since this condition is also sufficient, the third set may be chosen in 6 ways.

Likewise the operators of the fourth set must form a term of the expansion of M, and each be distinct from a_3, b_3, a_3b_3. But in the expansion of a determinant of the third order, exactly two terms have no element in common with a given term. One of the two being chosen as the fourth set, the other must be taken as the fifth set.

But the order of the sets is immaterial. Hence

$$N_2=(35.16.6.2)\div 5!=56.$$

3. Let next $n=3$. The number of subgroups of order 8 of G_{64} is

$$(63.62.60)\div(7.6.4)=5.9.31.$$

As our first set we may take the 7 operators $\neq$I of any G_8:

$$(2)\qquad \text{I},\ a,\ b,\ c,\ ab,\ ac,\ bc,\ abc.$$

For brevity we denote the set by (a, b, c), giving generating operators. For the second set (A, B, C), we may take as A any operator of G_{64} different from the 8 given by (2); as B, any operator of G_{64} different from the 16 given by (2) and their products by A; as C, any operator different from the 32 given by (2) and their products by A, B, AB. These necessary conditions are also sufficient. Indeed, a rectangular table for G_{64} may be formed by taking the operators (2) in the first row and the operators of (A, B, C) as the multipliers. Hence, we may choose the second set in $(56.48.32)\div(7.6.4)=8^3$ ways, since there are 7.6.4 triples in the set which may be taken as generators.

The 7 operators of the third set (a_3, b_3, c_3) must be products a_1a_2, in which the 7 a_1's form a permutation of the elements of the first set, the 7 a_2's a permutation of those of the second set. For if two a_1's were equal, the product of the corresponding operators would belong to the second set. These conditions are not sufficient, in contrast to $n=2$. Let $a_3=a_1a_2, b_3=a_1'a_2', c_3=a_1''a_2''$. In view of $a_3b_3c_3$, we must have $a_ia_i'a_i''\neq$I for $i=1$ and 2. *The necessary and sufficient conditions for a third set* $(a_1a_2, a_1'a_2', a_1''a_2'')$ *are that* a_1, a_1', a_1'' *be three distinct elements of the first set such that* $a_1a_1'a_1''\neq$I, *and that* a_2, a_2', a_2'' *be*

three distinct elements of the second set such that $a_2 a_2' a_2'' \neq I$. Thus a_1 may be chosen in 7 ways, a_1' in 6 ways, a_1'' in 4 ways. Similarly for a_2, a_2', a_2''. Hence the third set may be chosen in $(7.6.4)^2 \div (7.6.4)$ ways.

By a suitable rearrangement of the 7 elements of the first set, we may fix the notation so that the first three sets are

$$(3) \qquad (a, b, c), \quad (A, B, C), \quad (aA, bB, cC).$$

From the resulting symmetry, these sets are not altered if we make any rearrangement of the 7 elements of the first set and the same rearrangement of the corresponding 7 elements of the second set. In particular, we may leave b unaltered, replace a by d, and c by e, where b, d, e generate the first set, making the corresponding changes in the second set. Hence we can throw a fourth set containing aB into one containing dB, where d is any one of the 6 elements (1) except I and b. Consider therefore the fourth sets containing aB. No one of them contains bA, since $abAB$ lies in the third set. No one contains abA; for, aB, abA, and their product bAB, would then occur, and hence also tcC, where t is a, b, or ab, whereas the products

$$acC.bAB, \quad bcC.abA, \quad abcC.aB$$

belong to the third set. Hence a fourth set with aB contains cA, acA, bcA, or $abcA$. But a fourth set with aB and cA can be thrown into one with aB and dA, where $d = ac$, bc, or abc, by replacing c by d, C by D, without altering a, b, A, B. A fourth set with aB and cA must be one of the two:

$$(4) \qquad aB, \ cA, \ acAB, \ abC, \ bBC, \ abcAC, \ bcABC;$$
$$(5) \qquad aB, \ cA, \ acAB, \ bcC, \ abcBC, \ bAC, \ abABC.$$

Hence the fourth set may be chosen in exactly 6.4.2 ways.

If in (5) we interchange the small and capital letters, and then permute b with c, B with C, we obtain (4). By these changes, the first and second sets (3) are interchanged, while the third set is unaltered. *By a suitable change of notation the first four sets may be written in the form*

$$(6) \qquad (a, b, c), \quad (A, B, C), \quad (aA, bB, cC), \quad (aB, cA, abC).$$

Next, there are exactly 12 sets having no element in common with one of the four sets (6). These are

$$(7) \qquad cB, \ abA, \ abcAB, \ acC, \ aBC, \ bcAC, \ bABC;$$
$$(8) \qquad abB, \ acA, \ bcAB, \ abcC, \ cBC, \ bAC, \ aABC;$$
$$(9) \qquad acB, \ abcA, \ bAB, \ bcC, \ abBC, \ aAC, \ cABC;$$
$$(10) \qquad bcB, \ bA, \ cAB, \ aC, \ abcBC, \ abAC, \ acABC;$$
$$(11) \qquad abcB, \ bcA, \ aAB, \ bC, \ acBC, \ cAC, \ abABC;$$

(12) $cB,\ bcA,\ bAB,\ abcC,\ abBC,\ aAC,\ acABC;$
(13) $abB,\ abcA,\ cAB,\ bcC,\ acBC,\ aAC,\ bABC;$
(14) $acB,\ bA,\ abcAB,\ bcC,\ abBC,\ cAC,\ aABC;$
(15) $acB,\ abA,\ bcAB,\ bC,\ abcBC,\ aAC,\ cABC;$
(16) $acB,\ abcA,\ bAB,\ aC,\ cBC,\ bcAC,\ abABC;$
(17) $bcB,\ abcA,\ aAB,\ acC,\ abBC,\ bAC,\ cABC;$
(18) $abcB,\ acA,\ bAB,\ bcC,\ aBC,\ abAC,\ cABC.$

But (12) has an element in common with each of the sets (7)—(18). This is also true if we start with (13), ..., or (18) instead of (12). Hence the fifth, sixth, ..., ninth sets must be chosen from (7)—(11). Inversely, no two of the sets (6)—(11) have an element in common, and hence give a solution of the problem. We have now proved that

$$N_3=(5.9.31)8^3(7.6.4)(6.4.2)5!\div 9!=2^{12}.3.5.31).$$

Theorem. *Each of the* $2^{12}.3.5.31$ *solutions of the problem may be derived by a suitable change of notation from the solution given by the nine sets* (6)—(11).

THE UNIVERSITY OF CHICAGO, *June, 1904.*

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

205. Proposed by G. B.M. ZERR, A. M., Ph. D., Parsons, W. Va.

Express in the form of radicals the roots of the equation:

$$x^{15}+15mx^{13}+90m^2x^{11}+275m^3x^9+450m^4x^7+378m^5x^5+140m^6x^3+15m^7x+2r=0.$$

I. Solution by A. H. HOLMES, Brunswick, Maine.

$x^{15}+15mx^{13}+90m^2x^{11}+275m^3x^9+459m^4x^7+378m^5x^5+140m^6x^3+15m^7x+2r=0.$

$\therefore (x^5+5mx^3+5m^2x)+3m^5(x^5+5mx^3+5m^2x)+2r=0.$

Put $x^5+5mx^3+5m^2x=y.$

$\therefore y^3+3m^5y+2r=0.$

By the well known method of solving cubics:

$y_1=[-r+\sqrt{(m^{15}+r^2)}]^{\frac{1}{3}}+[-r-\sqrt{(m^{15}+r^2)}]^{\frac{1}{3}}=t_1+t_2=R_1,$

$y_2=-\tfrac{1}{2}(t_1+t_2)+\tfrac{1}{2}\sqrt{-3}(t_1-t_2)=R_2,$

$y_3=-\tfrac{1}{2}(t_1+t_2)-\tfrac{1}{2}\sqrt{-3}(t_1-t_2)=R_3,$

$\therefore x^5+5mx^3+5m^2x=R_1$ or R_2 or $R_3.$

Taking the first of these values,

$$r_1=\left(\frac{R_1}{2}+\sqrt{m^5+\frac{R_1^2}{4}}\right)^{\frac{1}{5}}+\left(\frac{R_1}{2}-\sqrt{m^5+\frac{R_1^2}{4}}\right)^{\frac{1}{5}}=s_1+s_2,$$

$x_2=es_1+e^4s_4$; $x_3=e^2s_1+e^3s_4$; $x_4=e^3s_1+e^2s_4$, and $x_5=e^4s_1+es_4$, when e represents the imaginary fifth root of unity.

The other ten values of x can be found by substituting R_2 and R_3 for R_1 above.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $x=y+z$.

$\therefore y^{15}+z^{15}+(yz+m)[15(y^{13}+z^{13})+(105yz+90m)(y''+z'')+(455y^2z^2+715myz+275m^2)(y^9+z^9)+(1365y^3z^3+2925my^2z^2+2025m^2yz+450m^3)(y^7+z^7)+(3003y^4z^4+7722my^3z^3+7128m^2y^2z^2+2772m^3yz+378m^4)(y^5+z^5)+(5005y^5z^5+14300my^4z^4+15400m^2y^3z^3+7700m^3y^2z^2+1750m^4yz+140m^5)(y^3+z^3)+(6435y^6z^6+19305my^5z^5+22275m^2y^4z^4+12375m^3y^3z^3+3375m^4y^2z^2+405m^5yz+15m^6)(y+z)]+2r=0.$

$\therefore y^{15}+z^{15}=-2r$, $yz=-m$. Let $y^{15}=a$, $z^{15}=b$.

$\therefore a+b=-2r$, $ab=-m^{15}$. $\therefore a$ and b are the roots of $t^2+2rt-m^{15}=0$.

Let β=an imaginary fifteenth root of unity and also let $a^{\frac{1}{15}}=c$, $b^{\frac{1}{15}}=d$.

$\therefore$ The roots are $c+d$, $\beta c+\beta^{14}d$, $\beta^{14}c+\beta d$, $\beta^2c+\beta^{13}d$, $\beta^{13}c+\beta^2d$, $\beta^3c+\beta^{12}d$, $\beta^{12}c+\beta^3d$, $\beta^4c+\beta^{11}d$, $\beta^{11}c+\beta^4d$, $\beta^5c+\beta^{10}d$, $\beta^{10}c+\beta^5d$, $\beta^6c+\beta^9d$, $\beta^9c+\beta^6d$, $\beta^7c+\beta^8d$, $\beta^8c+\beta^7d$.

Also solved by J. Scheffer.

206. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

The product of a certain pair of roots of $x^4+ax^3+bx^2+amx+m^2=0$, is equal to the product of the remaining pair.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let x_1, x_2, x_3, x_4 be the roots. Then $x_1+x_2+x_3+x_4=-a$; $x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4=-am$; $x_1x_2x_3x_4=m^2$.

$\therefore m(x_1+x_2+x_3+x_4)=x_1x_2(x_3+x_4)+x_3x_4(x_1+x_2)=-am$.

$\therefore x_1x_2=x_3x_4=m$.

The same is true if $x_1x_3=x_2x_4=m$, $x_1x_4=x_2x_3=m$.

II. Solution by J. SCHEFFER, Kee Mar College, Hagerstown, Md.

From the theory of equations, we have $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-am$, or $\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=-am$, but $\gamma\delta=\alpha\beta$.

$\therefore \alpha\beta(\alpha+\beta+\gamma+\delta)=-am$. Since $\alpha+\beta+\gamma+\delta=-a$, $-\alpha\beta a=-am$.

$\therefore \alpha\beta=m=\gamma\delta$.

III. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

$$\Pi(\alpha\beta-\gamma\delta)=\alpha\beta\gamma\delta(\Sigma a)^2-(\Sigma a\beta\gamma)^2=m^2(-a)^2-(-am)^2=0.$$

Hence the product of any one pair of the roots equals that of the remaining pair.

Solved by A. H. Holmes by actual computation of the roots.
Also solved by the Proposer.

207. Proposed by A. J. PAULSEN, San Francisco, Cal.

Solve $x^4+y^4=2x^2y^2$; $x+y=a$.

Solution by EDWIN L. RICH, Lehigh University, and A. H. HOLMES, Brunswick, Maine.

Writing the first of these equations in the form $(x+y)^2(x-y)^2=0$, it is easily seen, since $x+y=a$, that the four values of x are ∞, $-\infty$, $\frac{1}{2}a$, $\frac{1}{2}a$; and those of y are $-\infty$, ∞, $\frac{1}{2}a$, $\frac{1}{2}a$.

Also solved by G. B. M. Zerr, G. W. Greenwood, M. E. Graber, and J. Scheffer.

GEOMETRY.

233. Proposed by S. F. NORRIS, Professor of Mathematics, Baltimore City College, Baltimore, Md.

If from any point on a circle circumscribed about a triangle perpendiculars are dropped to the sides of the triangle, the feet of these perpendiculars lie on a line. [Ashton's *Plane and Solid Analytic Geometry*, page 87, 11th example].

Remark by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

See solution of Geometry Problem number 184, August-September, 1902.

234. Proposed by M. E. GRABER, A. B., Instructor in Mathematics and Physics in Heidelberg University, Tiffin, Ohio.

Find the curve which is reciprocal to a circle and define it as a locus.

I. Solution by the PROPOSER.

If A be the center of the given circle, P any doint on it; x and a lines corresponding to P and A; and X the point ax, $AP=\dfrac{OA.OX}{Ox}\sin ax$. Denoting OX by z the ratio $\sin ax:\sin zx$ is constant. The reciprocal to a circle is then the envelope of a line x which divides the angle between a fixed line a and a variable line z passing through a fixed point O, into parts whose sines are in a constant ratio. Defined as a locus, the reciprocal curve to a circle is the path of a point which moves so that its distance from a fixed point varies as its distance from a fixed straight line. [Lachlan's *Modern Pure Geometry*.]

II. Solution by G. W. GREENWOOD, M. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

See Russell's *Pure Geometry*, Chapter VIII, §11, or Salmon's *Conic Sections*, §308. The following is an analytic solution. Call the center of the given circle C and its radius a. Call the center of the circle of reciprocation O. De-

note CO by d; at any point P of the given circle draw a tangent to it, and through O draw a perpendicular to this circle, meeting it at N. Denote PCO by θ. The locus of N is therefore $r=a-d\cos\theta$, and the inverse of N with respect to the circle whose center is O, that is, the reciprocal of the given circle has for its equation

$$\frac{k^2}{r}=a-d\cos\theta.$$

Hence the required locus is a conic, whose focus is O, and eccentricity d/a. It is therefore an ellipse, an hyperbola, or a parabola, according as $d<$, $=$, or $>a$.

Also solved by G. B. M. Zerr.

235. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Any point on an ellipse is joined to the corners of an inscribed square. Find the anharmonic ratios of the pencil so formed.

I. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

We can show that the center of the square is the center of the ellipse and that the sides are parallel to its axes. Now anharmonic ratios are unaltered by projection. Hence project the ellipse into a circle whose diameter is $2b$, where a, b are the semi-major axis and semi-minor axis, respectively. We have a circle with an inscribed rectangle $ABCD$ where $\frac{AB}{AD}=\frac{a}{b}$. Consider the pencil $P(ABCD)$ where P is any point on the circle. The anharmonic ratio is

$$\frac{\sin APB.\sin CPD}{\sin CPB.\sin APD}.$$

Calling $\angle APB$, θ, we have

$$\frac{\sin\theta.\sin\theta}{\cos\theta.\cos\theta}=\tan^2\theta.$$

Now $\tan\theta=\frac{AB}{AD}=\frac{a}{b}$. Hence the anharmonic ratio is $\frac{a^2}{b^2}$.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let $ABCD$ represent the inscribed square, P any point. Let PD and PC intersect AB at M and N. Let $2c=$length of square, and $a^2y^2+b^2x^2=a^2b^2$ the equation of the ellipse; and x_1, y_1 coördinates of point P. The coördinates of A, B, C, D are respectively (c, c), $(c, -c)$, $(-c, -c)$, $(-c, c)$. Hence the equation of PD is $y-c=\frac{y_1-c}{x_1+c}(x+c)$; of PC, $y+c=\frac{y_1+c}{x_1+c}(x+c)$; of AB, $x=c$.

$\therefore$ The coördinates of point $M=\left(c, c+\frac{(y_1+c)2c}{x_1+c}\right)$; of N, $\left(c, -c+\frac{(y_1+c)2c}{x_1+c}\right)$;

$\therefore AM=\frac{2c(c-y_1)}{c+x_1}$, $BN=\frac{2c(c+y_1)}{c+x_1}$, $AB=2c$, $MN=\frac{2c(x_1-c)}{x_1+c}$.

$$\therefore \text{An anharmonic ratio } \frac{AM}{MN}:\frac{AB}{BN}=\frac{c^2-y_1^2}{x_1^2-c^2}; \text{ but } c^2=\frac{a^2b^2}{a^2+b^2}.$$

$$\therefore \text{An anharmonic ratio}=\frac{b^2}{a^2}\cdot\frac{(a^2+b^2)x_1^2-a^2b^2}{(a^2+b^2)x_1^2-a^2b^2}=\frac{b^2}{a^2}.$$

Also solved by G. B. M. Zerr.

CALCULUS.

181. Proposed by S. F. NORRIS, Baltimore, Md.

Integrate $dy=\frac{x^2dx}{1+x^4}$. [From Olney's *Integral Calculus*, page 116, third example, second part].

Solution by G. W. GREENWOOD, M. A.; M. E. GRABER, M. A., and G. B. M. ZERR, A. M., Ph. D.

$$\frac{x^2}{1+x^4}=\frac{1}{2\sqrt{2}}\left[\frac{x}{x^2-x\sqrt{2}+1}-\frac{x}{x^2+x\sqrt{2}+1}\right]$$

$$=\frac{1}{4\sqrt{2}}\left[\frac{(2x-\sqrt{2})+\sqrt{2}}{x^2-x\sqrt{2}+1}-\frac{(2x+\sqrt{2})-\sqrt{2}}{x^2+x\sqrt{2}+1}\right]$$

$$=\frac{1}{4\sqrt{2}}\left[\frac{2x-\sqrt{2}}{x^2-x\sqrt{2}+1}-\frac{2x+\sqrt{2}}{x^2+x\sqrt{2}+1}\right]$$

$$+\tfrac{1}{4}\left[\frac{1}{x^2-x\sqrt{2}+1}+\frac{1}{x^2+x\sqrt{2}+1}\right].$$

The required integral is therefore

$$=\frac{1}{4\sqrt{2}}[\log(x^2-x\sqrt{2}+1)-\log(x^2+x\sqrt{2}+1)]$$

$$+\frac{\sqrt{2}}{4}[\tan^{-1}(\sqrt{2}x-1)+\tan^{-1}(\sqrt{2}x+1]$$

$$=\frac{1}{4\sqrt{2}}\log\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}+\frac{\sqrt{2}}{4}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}.$$

Also solved by J. Scheffer.

182. Proposed by A. H. HOLMES, Brunswick, Maine.

Evaluate $\int_0^{\frac{1}{2}\pi} d\theta/\sqrt{[1+\sin^2\theta(1-4\cos\theta)]}$.

Solution by S. A. COREY, Hiteman, Iowa.

$$\int_0^{\frac{1}{2}\pi} d\theta \sqrt{[1+\sin^2\theta(1-4\cos\theta)]} \equiv \int_0^{\frac{1}{2}\pi} d\theta \sqrt{(1\tfrac{1}{2}-\cos\theta-\tfrac{1}{2}\cos 2\theta+\cos 3\theta)} \quad \ldots\ldots\ldots\ldots (1).$$

The value of the second member of (1) is easily computed by aid of the formula

$$f(x)-f(0)=\frac{x}{m.2}\left\{[f'(x)+f'(0)]+2\left[f\left(\frac{x}{m}\right)+f'\left(\frac{2x}{m}\right)+f'\left(\frac{3x}{m}\right)+\ldots\ldots\right.\right.$$

$$\left.\left.\ldots\ldots+f'\left(\frac{m-1}{m}x\right)\right]\right\}-\frac{B_1 x^2}{m^2.2!}[f'(x)-f'(0)]+\frac{B_2 x^4}{m^4.4!}[f^{iv}(x)-f^{iv}(0)]+\ldots\ldots$$

$$\ldots\ldots+(-1)^n\frac{B^n x^{2n}}{m^{2n}.(2n)!}[f^{2n}(x)-f^{2n}(0)]+\ldots\ldots$$

B_1, B_2, ..., being Bernoulli's numbers.

Taking $m=5$, and $n=2$, the value of the definite integral is found to be 1.17066. For greater accuracy, larger values of m and n may be taken.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $\sin^2\theta(1-4\cos\theta)=x$. Then by expanding the expression we get

$$(1+x)^{\frac{1}{2}}=1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3-\tfrac{5}{128}x^4+\tfrac{7}{256}x^5-\ldots\ldots.$$

$$\int_0^{\frac{1}{2}\pi} d\theta=\tfrac{1}{2}\pi=1.5708; \qquad \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} x\,d\theta=\tfrac{1}{8}\pi-\tfrac{2}{3}=-.2740;$$

$$-\tfrac{1}{8}\int_0^{\frac{1}{2}\pi} x^2 d\theta=-\frac{11\pi}{128}=-.0699; \qquad \tfrac{1}{16}\int_0^{\frac{1}{2}\pi} x^3 d\theta=\frac{35\pi}{512}-\tfrac{59}{256}=-.0194;$$

$$-\tfrac{5}{128}\int_0^{\frac{1}{2}\pi} x^4 d\theta=\tfrac{21}{704}\tfrac{5}{2}-\frac{2975\pi}{32768}=-.0138;$$

$$\tfrac{7}{256}\int_0^{\frac{1}{2}\pi} x^5 d\theta=\frac{16401\pi}{131072}-\frac{54761}{137280}=-.0059;$$

$$\therefore \int_0^{\frac{1}{2}\pi} \sqrt{[1+\sin^2\theta(1-4\cos\theta)]}\,d\theta=1.1858-.$$

Also solved by the Proposer.

MECHANICS.

170. Proposed by ELISHA S. LOOMIS, Ph. D., Berea, Ohio.

Two angles of iron, A_1CD and A_1CA_2, move freely on a pivot at C. Rods B_1A_1 and B_1A_2 are attached respectively at A_1 and at some point A_2 so that

when B_1 moves along the rod CR, which is perpendicular to A_1A_4, CD and CA_3 shall coincide in position with CE which is perpendicular to rod KR. When angle A_1CD is 135° find CA_3 in terms of CA_1.

Also find the following:

a). That value of CB_1 which will require least effort exerted at B_1 to cause CA_3 to take the position CA_4.

b). That value of CB_1 which will cause B_2A_2, if produced, to pass through the point A_1.

c). As CB_1 varies in value, what is the locus of the intersection of A_1B_1 and A_2B_2? Of B_1A_3 and B_2A_4?

d). Suppose angle A_1CD to be any other angle than 135°, then find CA_3 in terms of CA_1.

Solution by the PROPOSER.*

Determination of the value of CA_3 in terms of CA_1 when angle $A_1CD=135°$.

a). In triangle CA_1B_1, let $CA_1=r$ and $CB_1=vr$. Then $A_1B_1=r\sqrt{(1+v^2)}$.

b). In triangle CA_2B_2, $CA_2=r$, $A_2B_2=r\sqrt{(1+v^2)}$, and angle $A_2CB_2=45°$. Then, by trigonometry, $\sin B_2=\dfrac{r\sin C}{r\sqrt{(1+v^2)}}=\dfrac{1}{\sqrt{2}\sqrt{(1+v^2)}}$.

$$\therefore CB_2A_2=\sin^{-1}\frac{1}{\sqrt{2}\sqrt{(1+v^2)}}.$$

$$\therefore CA_2B_2=135°-\sin^{-1}\frac{1}{\sqrt{2}\sqrt{(1+v^2)}}.$$

$$\therefore CB_2=\frac{r\sqrt{(1+v^2)}\sin\left(137°-\sin^{-1}\dfrac{1}{\sqrt{2}\sqrt{(1+v^2)}}\right)}{\sin 45°}$$

$$=r\sqrt{2}\sqrt{(1+v^2)}\sin\left(135°-\sin^{-1}\frac{1}{\sqrt{2}\sqrt{(1+v^2)}}\right)$$

$$=kr, \text{ in which } k=\sqrt{2}\sqrt{(1+v^2)}\sin\left(135°-\sin^{-1}\frac{1}{\sqrt{2}\sqrt{(1+v^2)}}\right).$$

c). In triangle CB_1A_3, $CB_1=vr$ and angle $B_1CA_3=135°$. Draw A_3H perpendicular to CK. Then $CA=A_3H$. Let $CA_3=xr$, and $CH=p=xr/\sqrt{2}$.

$$\therefore (B_1A_3)^2=(CB_1)^2+(CA_3)^2+2CB_1\times CH=(vr)^2+(xr)^2+\sqrt{2}\,xvr^2$$
$$=r^2(v^2+x^2+\sqrt{2}\,vx).$$

d). In triangle B_2CA_4, $CA_4=CA_3=xr$, $B_1A_3=B_2A_4$, $CB_2=kr$, and angle $B_2CA_4=90°$. $(B_2A_4)^2=(B_2C)^2+(CA_4)^2=r^2(k^2+x^2)$.

$$\therefore r^2(v^2+x^2+\sqrt{2}\,vx)=r^2(k^2+x^2).\quad \therefore v^2+\sqrt{2}\,vx=k^2.\quad \therefore \sqrt{2}\,vx=k^2-v^2.$$

$$\therefore x=\frac{k^2-v^2}{\sqrt{2}\,v}.\qquad \therefore xr=\frac{r\{[\sqrt{2}\sqrt{(1+v^2)}\sin(135°-\sin^{-1}\dfrac{1}{\sqrt{2}\sqrt{(1+v^2)}})]^2-v^2\}}{\sqrt{2}\,v}.$$

Also solved by G. B. M. Zerr.

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*Dr. Loomis writes us that the determination of CA_3 in terms of CA_1 is needed in a certain mechanism for producing dissolving views in a magic lantern. Ed. E.

PROBLEMS FOR SOLUTION.

ALGEBRA.

211. Proposed by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Prove that $p-qx$ and $q-px$ tend to equality as x dimishes to zero, but yet that their limits are not equal. [Edwards' *Differential Calculus*, p. 7, ex. 10.]

212. Proposed by F. P. MATZ, Ph. D., Sc. D.

Solve $\dfrac{x^2+2x+3}{x^2-2x+3}+\dfrac{x^2-2x+3}{x^2+2x+3}=\dfrac{10}{3}$.

213. Proposed by F. P. MATZ, Ph. D., Sc. D.

Find the two roots of the equation $x^5-209x+56=0$, whose product is unity.

214. Proposed by Editor EPSTEEN.

How many terms are there in the expansion of $(x_1+x_2+\ldots\ldots+x_n)^n$?

GEOMETRY.

239. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Divide the sides of a triangle ABC internally in P, Q, R, so that $BP/PC=CQ/QA=AR/RB$. QR cuts BC externally in S. Show that BS is to CS in the duplicate ratio of CP to PB.

240. Proposed by B. F. BURLESON.

The points P_1, P_2, P_3 in the perpendicular BD of an isosceles triangle with equal angles at A and C are at the intersection of the three perpendiculars of the triangles, the center of its inscribed, and the center of its circumscribed circles, respectively. The distance from P_1 to P_2 is $m=16$ rods, and the distance from P_2 to P_3 is $n=17$ rods. Required the radii R and r of the triangle's circumscribed and inscribed circles, the perpendiculars $BD=P_b$, the base $AC=2b$, and one of the equal sides as $AB=a$.

241. Proposed by Editor EPSTEEN.

If two conics have each double contact with a third, their chords of contact with that conic, and two of the lines through their common points, will meet in a point and form a harmonic pencil.

242. Proposed by the late MARCUS BAKER.

In a trapezoid $ABCD$, upper base $BC=a$, lower base $AD=b$, a line CP is drawn from vertex C to any point P in the base AD, such that $PD=mb$. The line CP intersects the diagonal BP in M and MN is drawn parallel to the bases meeting CD in N; then is $MN=\dfrac{abm}{a+bm}$.

CALCULUS.

185. Proposed by A. H. HOLMES, Brunswick, Maine.

Required the perpendicular height of a right cone, radius of base being unity, such that the maximum ellipse that can be cut from the cone shall equal the base of the cone in area.

186. Proposed by Editor EPSTEEN.

Evaluate $\int_0^\infty \frac{\sin my}{y} dy$, $\int_0^\infty \frac{\cos my}{y} dy$.

GROUP THEORY.

6. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Show that the binary substitutions on ξ_1, η_1, the binary substitutions on ξ_2, η_2, and $(\xi_1\xi_2)(\eta_1\eta_2)$ generate a maximal subgroup of the quaternary abelian group.

MISCELLANEOUS.

146. Proposed by F. P. MATZ, Ph. D., Sc. D.

Given $\left.\begin{matrix} a\cos\alpha + b\sin\alpha = c \\ a\cos\beta + b\sin\beta = c \end{matrix}\right\}$ to prove that

$$\sin(\alpha+\beta) = \frac{2ab}{a^2+b^2}, \text{ and } \cot\alpha + \cot\beta = \frac{2ab}{c^2-a^2}.$$

NOTES.

Mr. H. R. Willard has been appointed instructor in Mathematics in the University of Maine.

Mr. C. A. Holden has been appointed assistant professor of Mathematics in Dartmouth College.

Dr. C. Gunderson has been appointed instructor in Mathematics in the Michigan Agricultural College.

Mr. C. H. Sisam has been appointed instructor in Mathematics at the U. S. Naval Academy, Annapolis.

Mr. W. D. Cairns has been promoted to an associate professorship in Mathematics at Oberlin College.

Prof. T. F. Nichols, of Hamilton College, has been promoted to a full professorship of Applied Mathematics.

Miss M. E. Sinclair and Mr. C. Havemeyer have been appointed instructors in Mathematics in the University of Nebraska.

Dr. George H. Hallett has been promoted to an assistant professorship of Mathematics in the University of Pennsylvania.

Mr. N. R. Wilson, lecturer in Mathematics in Wesley College, Winnipeg, Manitoba, is doing advanced work at the University of Chicago.

Prof. T. E. Holgate of the department of Mathematics of Northwestern University, has been appointed acting President of the University.

Dr. E. M. Blake, instructor in Mathematics in the University of California, has accepted the chair of Mathematics in the University of Arizona.

Professor Vining of Brandon College, Brandon, Manitoba, has been granted a leave of absence for two years. His place is temporarily filled by Dr. H. E. Jordan.

BOOKS.

Lehrbuch der Differenzenrechnung by D. Seliwanoff. B. G. Teubner, Leipzig, 1904. 92 pp.

At the request of the well known Leipzig publishers, B. G. Teubner, the author has elaborated his article on Finite Differences in the *Encyklopaedie der Mathematischen Wissenschaften*, Vol. 1, pp. 918-937, to the dimensions of a book. Thus, while the encyclopaedia gives but one page to the approximate evaluation of definite integrals and four pages to the subject of difference equations, the book devotes six pages to the former and twenty-nine pages to the latter. The gain in perspicuity over the encyclopaedia article is therefore considerable and indeed, although the author omits certain questions which might well be taken up, the subjects which he treats are presented in a delightfully clear and simple manner.

The books of Boole and Markoff are more complete, but this work of Seliwanoff should be regarded not as a handbook for one who is familiar with the subject, but as a text book for the beginner who desires to learn the *technique* of computation, such as the methods of interpolation, construction of tables, estimation of unavoidable errors.

Part I devotes thirty-two pages to the subject of Differences. After developing some of the most important general theorems in Chapter 1, the question of Interpolation is taken up in Chapter 2 where the author considers exact and approximate interpolation, computation of the roots of numerical equations, and the computation of logarithms and antilogarithms. The methods of evaluation of definite integrals in Chapter 3 are all very elementary, culminating with Simpson's Formula.

In Part II, Chapter 1 treats the subject of Indefinite and Definite Summation. According to the conventional usage the symbol Σ is employed as the Calculus analog of the sign of integration $\int$. In the opinion of the reviewer the symbol S would serve the purpose somewhat better. First, the use of the Greek letter as a functional symbol, where the (finite) integration is not possible, is a departure from its recognized meaning in various domains of analysis; second, the letter S corresponds somewhat closer to the

long s of the integral calculus, and third, the Σ would remain open to indicate a summation of integrals. Chapter 2 develops in an elementary manner the Jacob Bernoulli function $\varphi_n(x)=\sum_0^x \frac{x^{n-1}}{(n-1)!}$, and some of its properties. Chapters 3, 4 are devoted to the Euler formula

$$F(a)=\frac{1}{h}\int_a^{a+h} F(\mu)d\mu+A_1[F(a+h)-F(a)]+A_2h^2F'(a+\vartheta h)$$

and to some of its important applications, such as the development of Sterling's formula

$$\lg(1.2.3.....x)=\lg\sqrt{(2\pi)}+(x+\tfrac{1}{2})\lg x-x-A_2\cdot\frac{1}{x}+A_4\cdot\frac{2!}{x}+.....$$

Part III takes up the subject of Difference Equations. In Chapter 1 the general equation

$$y_{x+n}=f(x,\ y_x,\ y_{x+1},\ y_{x+2},\ .....,\ y_{x+n-1})$$

and the linear equation

$$y_{x+n}+p_1y_{x+n-1}+p_2y_{x+n-2}+.....+p_{n-1}y_{x+1}+p_ny_x=Q_x$$

are discussed very briefly. The second chapter is devoted to the linear equation of the first order and to the interesting application of expanding $\cos xt$ according to powers of $\cos t$. Chapter 3 treats in somewhat greater detail the linear difference equation with constant coefficients. Many of the questions of interest from the theoretical standpoint as well as in the higher applications, are not touched upon. We are indebted to the author for an excellent elementary text book on a fascinating and important subject. SAUL EPSTEEN.

Versicherungsmathematik. Von Alfred Loewy. Leipzig, Sammlung Göschen, 1903. 145 pp.

In publishing this book the Sammlung Göschen has certainly followed out successfully its expressed policy of giving to the public a brief, but clear and up-to-date development of one of the most interesting applications of mathematical theory. While the reader who is unacquainted with the subject of life insurance might find Professor Loewy's exposition somewhat too condensed, anyone with a knowledge of elementary algebra who has some acquaintance with the business aspect of the subject cannot fail to appreciate the value of this little pocket edition which contains in its 145 pages the development of all the important formula needed by the actuary.

While one recognizes at once the meaning of many of the words such as Nettoprämie =net premium, Sterblichkeitstafel=mortality table, the significance of some of the German expressions, of which a glossary of fourteen follows, is not at all evident. Indeed, a few are not to be found in the average German-English dictionary and their meaning must be learned from the context.

Zinsfuss=rate of interest.
Zinsezins=compound interest.
Barwert=present value.
Seibreute=annuity.
Erlebensversicherung=endowment.
Postnumerando=payable subsequently.
Pränumerando=payable in advance.
Rückversicherung=reinsurance.
Bruttoprämie=gross premium.
Prämienrückgewähr=return (of part or whole) of premium.
Rückkaufpreis=surrender value.
Passiva=liabilities.
Aktiva=assets.
Karenzzeit=period of deferred insurance (a policy payable at death, provided death is not within m years, is spoken of as a policy with m-jähriger karenzzeit).

Every time a conventional symbol is introduced its number (Roman) is printed in the margin, thus enabling the reader to find easily the meaning of the symbols in any formula under consideration. As might be expected, the notation is not that of the Actuarial Society of America; the author follows, with some exceptions, the Universal Standard Notation of the International Congress of Actuaries of 1895.

The fallacy involved in assessment life insurance is now well understood and Loewy therefore dismisses this subject with a brief and unfavorable mention.

On page 92 it is stated that a well known German company computes the gross premium A_x' of a one payment life policy by means of the formula $A_x' = \frac{21}{20} A_x$, where A_x = net premium. In America the "loading" is higher than this, but on the other hand life policies are "participating" and draw dividends.

In the case of entire life policies, yearly payments, the formula is $P_x' = 1.24 P_x$ for ages exceeding 34 years, making a loading of 24%. In America the loading is sometimes less and sometimes more than this, varying with the different companies. For non-participating policies it is much less. It is not calculated as a percentage of the net premium however. In the case of half yearly payments the Germans add one per cent. of the yearly net premium; in case of quarterly payments two per cent. is added.

The American law requires an insurance company to have on hand at the end of the first year, when the expenses are heaviest, the full reserve according to a mortality table in which no account is taken of the gain due to selection by medical examination. The large and wealthy companies meet this requirement by a bookkeeping device, transferring from the surplus, enough to meet the legal requirements, to the first year's reserve. The smaller companies are compelled however to resort to the following plan: If the contract is for a twenty payment life policy, they issue a one year term policy, the regular insurance commencing one year later and extending over nineteen years. Actuaries are divided in opinion at present on the moral point involved in this device; although unjust from the technical point of view it is difficult to see how a new company could come into existence in any other way. The German law is more flexible, allowing the company to draw upon the first year's reserve, for expenses, to the extent of $12\frac{1}{2} M$. per thousand marks of insurance.

Experience shows that the Gompertz-Makeham formula gives the most satisfactory way of graduating mortality tables, and it is the method in general use at the present time. For this reason the reviewer regards the author's treatment as too brief and condensed. The mere statement of the result in the integrated form with the specification of the values of the constants is not sufficient for one who is reading it for the first time.

A very interesting feature of the book is the insight given by it into the conditions of German life which help one to understand why deferred temporary annuities and endowments are more popular in Germany than in America. The former are often purchased for the purpose of defraying the expenses of a son at the University. The latter, besides being a provision for old age, are also purchased for the purpose of providing a daughter with her "aussteuer" and a son with the expenses of the one year of military service. According to the German law those who pass the "secondär" examination and are willing to defray their own expenses are exempted from two of the required three years of service.

The book seems to be free from misprints and other errors. By careful condensation the author has reduced it to pocket size. It will be found a very useful little hand book.

SAUL EPSTEEN.

Grammar School Algebra. By David Eugene Smith, Ph. D., Professor of Mathematics in Teachers College, Columbia University, New York. 12mo. Cloth. vi + 154 pp. Price, 50 cents. Boston and Chicago: Ginn & Co.

This book introduces the student of Arithmetic to the mysteries of Algebra with ease and simplicity. It begins by showing the uses of Algebra by employing such practical applications as are within the range of the knowledge of the student. The work includes linear equations with two unknown quantities, and easy quadratics. Abundant oral and written exercises are provided.

B. F. F.

Elements of the Differential and Integral Calculus, with applications. By William Anthony Granville, Ph. D., Instructor in Mathematics in the Sheffield Scientific School of Yale University. 8vo. Semi-flexible Cloth, xiv + 462 pages. Price, $2.50. Boston and Chicago: Ginn & Co.

This Calculus, based on the Method of Limits, is the first of a series of mathematical text books to be published by Ginn & Co., and to be edited by Percey F. Smith, Professor of Mathematics in the Sheffield Scientific School of Yale Uuiversity.

The book is designed for the use of students in colleges and engineering schools. At the beginning of the book is given a large collection of elementary formulas from Algebra, Geometry, Trigonometry, and Analytical Geometry, serving to refresh the mind of the beginner with many of the implements with which his further progress is executed. Following the establishment of the fundamental principles, the utility of the Calculus is illustrated by numerous simple examples taken from Geometry and Mechanics. The figures throughout the book are excellent, and the typography is first class. B. F. F.

Where Did Life Begin? A brief inquiry as to the probable place of beginning and the natural courses of the migration therefrom of the Flora and Fauna of the Earth. A Monograph. By Gilbert H. Schribner. New Edition. 8vo. Cloth, xiii + 75 pages. New York: Charles Scribner's Sons.

In this little volume, the author puts forth the view and substantiates the same with fairly convincing argument that the beginning of all life occurred at the poles. The book first appeared in 1883. Since then, modern scientific research has greatly strengthened the argument herein contained. B. F. F.

Lectures on the Calculus of Variations. By Harris Hancock, Ph. D., Dr. Sc. (Paris), Professor of Mathematics in the University of Cincinnati. Large 8vo, Paper Cover, xvi + 292 pages.

In these lectures the author informs us that he has attempted to give a connected and simple treatment of Weierstrass' Theory of the Calculus of Variations. The student is introduced to the study of the subject by considering simple problems illustrating the connection between the Calculus of Variations and the Theory of Maxima and Minima.

The second part of the work beginning with Chapter XIII treats of the Theory of Relative Maxima and Minima in which the isoperimetrical problems are considered. This treatise constitutes a very important contribution to the literature of the subject. B. F. F.

Text-book of General Physics for High Schools and Colleges. By Joseph S. Ames, Ph. D., Professor of Physics and Director of the Physical Laboratory in the Johns Hopkins University. 8vo. Cloth, 768 pages. Price, $3.50. New York, Cincinnati, and Chicago: The American Book Co.

This book is not a revision of the author's previous text book, entitled the *Theory of Physics*. It is an entirely new book containing scientific discussions of the most recent discoveries in Physics, and clear and logical statements of the theory and experimental facts upon which the science is based.

The book is divided into the following sections: Mechanics and Properties of Matter, Heat, Vibrations and Waves, Light, Magnetism and Electricity, and each subject is treated with the clearness, thoroughness, and scientific accuracy characteristic of Professor Ames' writing. B. F. F.

ERRATA.

Page 195, line 5 from bottom, for G_4 read G_{24}.
Page 196, line 9, read $a_0\beta_0 + a_1\beta_1 + a_0\beta_1 + a_1\beta_0 + a_0a_1 + \beta_0\beta_1 = b$.
Page 196, line 11, for $(\beta_1 + \beta_1)$ read $(a_1 + \beta_1)$.

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THE AMERICAN MATHEMATICAL MONTHLY.

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THE TRANSCENDENCE OF π AND e.

By DR. OSWALD VEBLEN, The University of Chicago.

§1. The proof that π is a transcendental number is ordinarily arranged as follows. If π should satisfy any algebraic equation, so would $\pi.\sqrt{-1}$. But it is well known that

$$e^{\pi.\sqrt{-1}}=-1 \qquad (A).$$

Hence if $\pi.\sqrt{-1}$ is one of the m roots $z_1, z_2, \ldots\ldots, z_m$, of an algebraic equation, we must have

$$(e^{z_1}+1)(e^{z_2}+1)\ldots\ldots(e^{z_m}+1)=0 \qquad (B)$$

since one of its factors is zero. On expanding (B) we obtain

$$c+e^{x_1}+e^{x_2}+\ldots\ldots+e^{x_n}=0 \qquad (C)$$

where $x_1, x_2, \ldots\ldots, x_n$ are the n roots of an algebraic equation and where c is a whole number not zero. The rest of the argument consists in showing that equation (C) is impossible.

The proof* that (C) is impossible is so difficult for most students that it

*The principal references in English on the subject of the transcendence of π and e seem to be the translation by W. W. Beman of the chapter on Transcendental Numbers in Weber's *Algebra* published in the *Bulletin of the American Mathematical Society*, Vol. 3 (1897), p. 174, and the translation by Beman and Smith of Klein's *Famous Problems of Elementary Geometry* (Ginn & Co., Boston). A good elementary treatment in the German language is that by Weber and Wellstein, *Encyclopädie der Elementarmathematik*, Vol. I, pp. 418-432. (B. G. Teubner, Leipzig).